

Cramér–Lundberg Asymptotics

In **risk theory**, the classical risk model is a **compound Poisson risk model**. In the classical risk model, the number of claims up to time $t \geq 0$ is assumed to be a **Poisson process** $N(t)$ with a Poisson rate of $\lambda > 0$; the size or amount of the i th claim is a nonnegative random variable X_i , $i = 1, 2, \dots$; $\{X_1, X_2, \dots\}$ are assumed to be independent and identically distributed with common distribution function $P(x) = \Pr\{X_1 \leq x\}$ and common mean $\mu = \int_0^\infty \bar{P}(x) dx > 0$; and the claim sizes $\{X_1, X_2, \dots\}$ are independent of the claim number process $\{N(t), t \geq 0\}$. Suppose that an insurer charges premiums at a constant rate of $c > \lambda\mu$, then the **surplus** at time t of the insurer with an initial capital of $u \geq 0$ is given by

$$X(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0. \quad (1)$$

One of the key quantities in the classical risk model is the **ruin probability**, denoted by $\psi(u)$ as a function of $u \geq 0$, which is the probability that the surplus of the insurer is below zero at some time, namely,

$$\psi(u) = \Pr\{X(t) < 0 \text{ for some } t > 0\}. \quad (2)$$

To avoid ruin with certainty or $\psi(u) = 1$, it is necessary to assume that the safety loading θ , defined by $\theta = (c - \lambda\mu)/(\lambda\mu)$, is positive, or $\theta > 0$. By using **renewal arguments** and conditioning on the time and size of the first claim, it can be shown (e.g. (6) on page 6 of [25]) that the ruin probability satisfies the following integral equation, namely,

$$\psi(u) = \frac{\lambda}{c} \int_u^\infty \bar{P}(y) dy + \frac{\lambda}{c} \int_0^u \psi(u-y) \bar{P}(y) dy, \quad (3)$$

where throughout this article, $\bar{B}(x) = 1 - B(x)$ denotes the tail of a distribution function $B(x)$.

In general, it is very difficult to derive explicit and closed expressions for the ruin probability. However, under suitable conditions, one can obtain some approximations to the ruin probability.

The pioneering works on approximations to the ruin probability were achieved by **Cramér** and **Lundberg** as early as the 1930s under the Cramér–Lundberg condition. This condition is to assume that there exists a constant $\kappa > 0$, called the **adjustment coefficient**, satisfying the following Lundberg equation

$$\int_0^\infty e^{\kappa x} \bar{P}(x) dx = \frac{c}{\lambda},$$

or equivalently

$$\int_0^\infty e^{\kappa x} dF(x) = 1 + \theta, \quad (4)$$

where $F(x) = (1/\mu) \int_0^x \bar{P}(y) dy$ is the **equilibrium distribution** of P .

Under the condition (4), the Cramér–Lundberg asymptotic formula states that if

$$\int_0^\infty x e^{\kappa x} dF(x) < \infty,$$

then

$$\psi(u) \sim \frac{\theta \mu}{\kappa \int_0^\infty y e^{\kappa y} \bar{P}(y) dy} e^{-\kappa u} \text{ as } u \rightarrow \infty. \quad (5)$$

If

$$\int_0^\infty x e^{\kappa x} dF(x) = \infty, \quad (6)$$

then

$$\psi(u) = o(e^{-\kappa u}) \text{ as } u \rightarrow \infty; \quad (7)$$

and meanwhile, the Lundberg inequality states that

$$\psi(u) \leq e^{-\kappa u}, \quad u \geq 0, \quad (8)$$

where $a(x) \sim b(x)$ as $x \rightarrow \infty$ means $\lim_{x \rightarrow \infty} a(x)/b(x) = 1$.

The asymptotic formula (5) provides an exponential asymptotic estimate for the ruin probability as $u \rightarrow \infty$, while the Lundberg inequality (8) gives an exponential upper bound for the ruin probability for all $u \geq 0$. These two results constitute the

2 Cramér–Lundberg Asymptotics

well-known Cramér–Lundberg approximations for the ruin probability in the classical risk model.

When the claim sizes are exponentially distributed, that is, $\bar{P}(x) = e^{-x/\mu}$, $x \geq 0$, the ruin probability has an explicit expression given by

$$\psi(u) = \frac{1}{1+\theta} \exp \left\{ -\frac{\theta}{(1+\theta)\mu} u \right\}, \quad u \geq 0. \quad (9)$$

Thus, the Cramér–Lundberg asymptotic formula is exact when the claim sizes are exponentially distributed. Further, the Lundberg upper bound can be improved so that the improved Lundberg upper bound is also exact when the claim sizes are exponentially distributed. Indeed, it can be proved under the Cramér–Lundberg condition (e.g. [6, 26, 28, 45]) that

$$\psi(u) \leq \beta e^{-\kappa u}, \quad u \geq 0, \quad (10)$$

where β is a constant, given by

$$\beta^{-1} = \inf_{0 \leq t < \infty} \frac{\int_t^{\infty} e^{\kappa y} dF(y)}{e^{\kappa t} \bar{F}(t)},$$

and satisfies $0 < \beta \leq 1$.

This improved Lundberg upper bound (10) equals the ruin probability when the claim sizes are exponentially distributed. In fact, the constant β in (10) has an explicit expression of $\beta = 1/(1+\theta)$ if the distribution F has a decreasing **failure rate**; see, for example [45], for details.

The Cramér–Lundberg approximations provide an exponential description of the ruin probability in the classical risk model. They have become two standard results on ruin probabilities in risk theory.

The original proofs of the Cramér–Lundberg approximations were based on **Wiener–Hopf methods** and can be found in Cramér [8, 9] and Lundberg [29, 30]. However, these two results can be proved in different ways now. For example, the **martingale** approach of Gerber [20, 21], Wald’s identity in [35], and the induction method in [24] have been used to prove the Lundberg inequality. Further, since the integral equation (3) can be rewritten as the following **defective renewal equation**

$$\psi(u) = \frac{1}{1+\theta} \bar{F}(u) + \frac{1}{1+\theta} \int_0^u \psi(u-x) dF(x), \quad u \geq 0, \quad (11)$$

the Cramér–Lundberg asymptotic formula can be obtained simply from the key renewal theorem for the solution of a defective renewal equation, see, for instance [17]. All these methods are much simpler than the Wiener–Hopf methods used by Cramér and Lundberg and have been used extensively in **risk theory** and other disciplines. In particular, the martingale approach is a powerful tool for deriving exponential inequalities for ruin probabilities. See, for example [10], for a review on this topic. In addition, the induction method is very effective for one to improve and generalize the Lundberg inequality. The applications of the method for the generalizations and improvements of the Lundberg inequality can be found in [7, 41, 42, 44, 45].

Further, the key renewal theorem has become a standard method for deriving exponential asymptotic formulae for ruin probabilities and related ruin quantities, such as the distributions of the **surplus** just before ruin, the deficit at ruin, and the amount of claim causing ruin; see, for example [23, 45].

Moreover, the Cramér–Lundberg asymptotic formula is also available for the solution to defective renewal equation, see, for example [19, 39] for details. Also, a generalized Lundberg inequality for the solution to defective renewal equation can be found in [43].

On the other hand, the solution to the defective renewal equation (11) can be expressed as the tail of a **compound geometric distribution**, namely,

$$\psi(u) = \frac{\theta}{1+\theta} \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta} \right)^n \bar{F}^{(n)}(u), \quad u \geq 0, \quad (12)$$

where $F^{(n)}(x)$ is the n -fold convolution of the distribution function $F(x)$. This expression is known as **Beekman’s convolution series**.

Thus, the ruin probability in the classical risk model can be characterized as the tail of a compound geometric distribution. Indeed, the Cramér–Lundberg asymptotic formula and the Lundberg inequality can be stated generally for the tail of a compound geometric distribution. The tail of a compound geometric distribution is a very useful probability model arising in many applied probability fields such as risk theory, **queueing**, and **reliability**. More applications of a compound geometric distribution in risk theory can be found in [27, 45], and among others.

It is clear that the Cramér–Lundberg condition plays a critical role in the Cramér–Lundberg approximations. However, there are many interesting claim size distributions that do not satisfy the Cramér–Lundberg condition. For example, when the moment generating function of a distribution does not exist or a distribution is **heavy-tailed** such as **Pareto** and **lognormal distributions**, the Cramér–Lundberg condition is not valid. Further, even if the **moment generating function** of a distribution exists, the Cramér–Lundberg condition may still fail. In fact, there exist some claim size distributions, including certain **inverse Gaussian** and generalized inverse Gaussian distributions, so that for any $r > 0$ with $\int_0^\infty e^{rx} dF(x) < \infty$,

$$\int_0^\infty e^{rx} dF(x) < 1 + \theta.$$

Such distributions are said to be medium tailed; see, for example, [13] for details.

For these medium- and heavy-tailed claim size distributions, the Cramér–Lundberg approximations are not applicable. Indeed, the asymptotic behaviors of the ruin probability in these cases are totally different from those when the Cramér–Lundberg condition holds. For instance, if F is a **subexponential distribution**, which means

$$\lim_{x \rightarrow \infty} \frac{\overline{F^{(2)}}(x)}{\overline{F}(x)} = 2, \quad (13)$$

then the ruin probability $\psi(u)$ has the following asymptotic form

$$\psi(u) \sim \frac{1}{\theta} \overline{F}(u) \text{ as } u \rightarrow \infty,$$

which implies that ruin is asymptotically determined by a large claim. A review of the asymptotic behaviors of the ruin probability with medium- and heavy-tailed claim size distributions can be found in [15, 16].

However, the Cramér–Lundberg condition can be generalized so that a generalized Lundberg inequality holds for more general **claim size distributions**. In doing so, we recall from the theory of **stochastic orderings** that a distribution B supported on $[0, \infty)$ is said to be new worse than used (NWU) if for any $x \geq 0$ and $y \geq 0$,

$$\overline{B}(x + y) \geq \overline{B}(x)\overline{B}(y). \quad (14)$$

In particular, an exponential distribution is an example of an NWU distribution when the equality holds in (14).

Willmot [41] used an NWU distribution function to replace the exponential function in the Lundberg equation (4) and assumed that there exists an NWU distribution B so that

$$\int_0^\infty (\overline{B}(x))^{-1} dF(x) = 1 + \theta. \quad (15)$$

Under the condition (15), Willmot [41] derived a generalized Lundberg upper bound for the ruin probability, which states that

$$\psi(u) \leq \overline{B}(u), \quad u \geq 0. \quad (16)$$

The condition (15) can be satisfied by some medium and heavy-tailed claim size distributions. See, [6, 41, 42, 45] for more discussions on this aspect. However, the condition (15) still fails for some claim size distributions; see, for example, [6] for the explanation of this case.

Dickson [11] adopted a truncated Lundberg condition and assumed that for any $u > 0$ there exists a constant $\kappa_u > 0$ so that

$$\int_0^u e^{\kappa_u x} dF(x) = 1 + \theta. \quad (17)$$

Under the truncated condition (17), Dickson [11] derived an upper bound for the ruin probability, and further Cai and Garrido [7] gave an improved upper bound and a lower bound for the ruin probability, which state that

$$\frac{\theta e^{-2\kappa_u u} + \overline{F}(u)}{\theta + \overline{F}(u)} \leq \psi(u) \leq \frac{\theta e^{-\kappa_u u} + \overline{F}(u)}{\theta + \overline{F}(u)}, \quad u > 0. \quad (18)$$

The truncated condition (17) applies to any positive claim size distribution with a finite mean. In addition, even when the Cramér–Lundberg condition holds, the upper bound in (18) may be tighter than the Lundberg upper bound; see [7] for details.

The Cramér–Lundberg approximations are also available for ruin probabilities in some more general risk models. For instance, if the claim number process $N(t)$ in the classical risk model is assumed to be a **renewal process**, the resulting risk model is called the **compound renewal risk model** or the **Sparre Andersen risk model**. In this risk model, interclaim

4 Cramér–Lundberg Asymptotics

times $\{T_1, T_2, \dots\}$ form a sequence of independent and identically distributed positive random variables with common distribution function $G(t)$ and common mean $\int_0^\infty \overline{G}(t) dt = (1/\alpha) > 0$. The ruin probability in the Sparre Andersen risk model, denoted by $\psi^0(u)$, satisfies the same **defective renewal equation** as (11) for $\psi(u)$ and is thus the tail of a compound geometric distribution. However, the underlying distribution in the defective renewal equation in this case is unknown in general; see, for example, [14, 25] for details.

Suppose that there exists a constant $\kappa^0 > 0$ so that

$$E(e^{\kappa^0(X_1 - cT_1)}) = 1. \quad (19)$$

Thus, under the condition (19), by the key renewal theorem, we have

$$\psi^0(u) \sim C_0 e^{-\kappa^0 u} \text{ as } u \rightarrow \infty, \quad (20)$$

where $C_0 > 0$ is a constant. Unfortunately, the constant C_0 is unknown since it depends on the unknown underlying distribution. However, the Lundberg inequality holds for the ruin probability $\psi^0(u)$, which states that

$$\psi^0(u) \leq e^{-\kappa^0 u}, \quad u \geq 0; \quad (21)$$

see, for example, [25] for the proofs of these results.

Further, if the claim number process $N(t)$ in the classical risk model is assumed to be a **stationary renewal process**, the resulting risk model is called the compound stationary renewal risk model. In this risk model, interclaim times $\{T_1, T_2, \dots\}$ form a sequence of independent positive random variables; $\{T_2, T_3, \dots\}$ have a common distribution function $G(t)$ as that in the compound renewal risk model; and T_1 has an equilibrium distribution function of $G_e(t) = \alpha \int_0^t \overline{G}(s) ds$. The ruin probability in this risk model, denoted by $\psi^e(u)$, can be expressed as the function of $\psi^0(u)$, namely

$$\psi^e(u) = \frac{\alpha\mu}{c} \overline{F}(u) + \frac{\alpha\mu}{c} \int_0^u \psi^0(u-x) dF(x), \quad (22)$$

which follows from conditioning on the size and time of the first claim; see, for example, (40) on page 69 of [25].

Thus, applying (20) and (21) to (22), we have

$$\psi^e(u) \sim C_e e^{-\kappa^0 u} \text{ as } u \rightarrow \infty, \quad (23)$$

and

$$\psi^e(u) \leq \frac{\alpha}{c\kappa^0} (m(\kappa^0) - 1) e^{-\kappa^0 u}, \quad u \geq 0, \quad (24)$$

where $C_e = (\alpha/c\kappa^0)(m(\kappa^0) - 1)C_0$ and $m(t) = \int_0^\infty e^{tx} dP(x)$ is the **moment generating function** of the claim size distribution P . Like the case in the Sparre Andersen risk model, the constant C_e in the asymptotic formula (23) is also unknown. Further, the constant $(\alpha/c\kappa^0)(m(\kappa^0) - 1)$ in the Lundberg upper bound (24) may be greater than one.

The Cramér–Lundberg approximations to the ruin probability in a risk model when the claim number process is a **Cox process** can be found in [2, 25, 38]. For the Lundberg inequality for the ruin probability in the Poisson **shot noise** delayed-claims risk model, see [3]. Moreover, the Cramér–Lundberg approximations to ruin probabilities in **dependent risk** models can be found in [22, 31, 33].

In addition, the ruin probability in the perturbed compound Poisson risk model with diffusion also admits the Cramér–Lundberg approximations. In this risk model, the surplus process $X(t)$ satisfies

$$X(t) = u + ct - \sum_{i=1}^{N(t)} X_i + W_t, \quad t \geq 0, \quad (25)$$

where $\{W_t, t \geq 0\}$ is a **Wiener process**, independent of the Poisson process $\{N(t), t \geq 0\}$ and the claim sizes $\{X_1, X_2, \dots\}$, with infinitesimal drift 0 and infinitesimal variance $2D > 0$.

Denote the ruin probability in the perturbed risk model by $\psi_p(u)$ and assume that there exists a constant $R > 0$ so that

$$\lambda \int_0^\infty e^{Rx} dP(x) + DR^2 = \lambda + cR. \quad (26)$$

Then Dufresne and Gerber [12] derived the following Cramér–Lundberg asymptotic formula

$$\psi_p(u) \sim C_p e^{-Ru} \text{ as } u \rightarrow \infty,$$

and the following Lundberg upper bound

$$\psi_p(u) \leq e^{-Ru}, \quad u \geq 0, \quad (27)$$

where $C_p > 0$ is a known constant. For the Cramér–Lundberg approximations to ruin probabilities in more general perturbed risk models, see [18, 37]. A review of perturbed risk models and the

Cramér–Lundberg approximations to ruin probabilities in these models can be found in [36].

We point out that the Lundberg inequality is also available for ruin probabilities in risk models with interest. For example, Sundt and Teugels [40] derived the Lundberg upper bound for the ruin probability in the classical risk model with a constant force of interest; Cai and Dickson [5] gave exponential upper bounds for the ruin probability in the Sparre Andersen risk model with a constant force of interest; Yang [46] obtained exponential upper bounds for the ruin probability in a discrete time risk model with a constant rate of interest; and Cai [4] derived exponential upper bounds for ruin probabilities in generalized discrete time risk models with dependent rates of interest. A review of risk models with interest and investment and ruin probabilities in these models can be found in [32]. For more topics on the Cramér–Lundberg approximations to ruin probabilities, we refer to [1, 15, 21, 25, 34, 45], and references therein.

To sum up, the Cramér–Lundberg approximations provide an exponential asymptotic formula and an exponential upper bound for the ruin probability in the classical risk model or for the tail of a compound geometric distribution. These approximations are also available for ruin probabilities in other risk models and appear in many other applied probability models.

References

- [1] Asmussen, S. (2000). *Ruin Probabilities*, World Scientific, Singapore.
- [2] Björk, T. & Grandell, J. (1988). Exponential inequalities for ruin probabilities in the Cox case, *Scandinavian Actuarial Journal* 77–111.
- [3] Brémaud, P. (2000). An insensitivity property of Lundberg’s estimate for delayed claims, *Journal of Applied Probability* 37, 914–917.
- [4] Cai, J. (2002). Ruin probabilities under dependent rates of interest, *Journal of Applied Probability* 39, 312–323.
- [5] Cai, J. & Dickson, D.C.M. (2003). Upper bounds for ultimate ruin probabilities in the Sparre Andersen model with interest, *Insurance: Mathematics and Economics* 32, 61–71.
- [6] Cai, J. & Garrido, J. (1999a). A unified approach to the study of tail probabilities of compound distributions, *Journal of Applied Probability* 36, 1058–1073.
- [7] Cai, J. & Garrido, J. (1999b). Two-sided bounds for ruin probabilities when the adjustment coefficient does not exist, *Scandinavian Actuarial Journal* 80–92.
- [8] Cramér, H. (1930). On the Mathematical Theory of Risk, Skandia Jubilee Volume, Stockholm.
- [9] Cramér, H. (1955). Collective Risk Theory, Skandia Jubilee Volume, Stockholm.
- [10] Dassios, A. & Embrechts, P. (1989). Martingales and insurance risk, *Stochastic Models* 5, 181–217.
- [11] Dickson, D.C.M. (1994). An upper bound for the probability of ultimate ruin, *Scandinavian Actuarial Journal* 131–138.
- [12] Dufresne, F. & Gerber, H.U. (1991). Risk theory for the compound Poisson process that is perturbed by diffusion, *Insurance: Mathematics and Economics* 10, 51–59.
- [13] Embrechts, P. (1983). A property of the generalized inverse Gaussian distribution with some applications, *Journal of Applied Probability* 20, 537–544.
- [14] Embrechts, P. & Klüppelberg, C. (1993). Some aspects of insurance mathematics, *Theory of Probability and its Applications* 38, 262–295.
- [15] Embrechts, P., Klüppelberg, C. & Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin.
- [16] Embrechts, P. & Veraverbeke, N. (1982). Estimates of the probability of ruin with special emphasis on the possibility of large claims, *Insurance: Mathematics and Economics* 1, 55–72.
- [17] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol. II, Wiley, New York.
- [18] Furrer, H.J. & Schmidli, H. (1994). Exponential inequalities for ruin probabilities of risk processes perturbed by diffusion, *Insurance: Mathematics and Economics* 15, 23–36.
- [19] Gerber, H.U. (1970). An extension of the renewal equation and its application in the collective theory of risk, *Skandinavisk Aktuarietidskrift* 205–210.
- [20] Gerber, H.U. (1973). Martingales in risk theory, *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker* 73, 205–216.
- [21] Gerber, H.U. (1979). *An Introduction to Mathematical Risk Theory*, S.S. Heubner Foundation Monograph Series 8, University of Pennsylvania, Philadelphia.
- [22] Gerber, H.U. (1982). Ruin theory in the linear model, *Insurance: Mathematics and Economics* 1, 177–184.
- [23] Gerber, H.U. & Shiu, F.S.W. (1998). On the time value of ruin, *North American Actuarial Journal* 2, 48–78.
- [24] Goovaerts, M.J., Kass, R., van Heerwaarden, A.E. & Bauwelinckx, T. (1990). *Effective Actuarial Methods*, North Holland, Amsterdam.
- [25] Grandell, J. (1991). *Aspects of Risk Theory*, Springer-Verlag, New York.
- [26] Kalashnikov, V. (1996). Two-sided bounds for ruin probabilities, *Scandinavian Actuarial Journal* 1–18.
- [27] Kalashnikov, V. (1997). *Geometric Sums: Bounds for Rare Events with Applications*, Kluwer Academic Publishers, Dordrecht.

- [28] Lin, X. (1996). Tail of compound distributions and excess time, *Journal of Applied Probability* **33**, 184–195.
- [29] Lundberg, F. (1926). in *Försäkringsteknisk Riskutjämn- ing*, F. Englund, A.B. Bobtryckeri, Stockholm.
- [30] Lundberg, F. (1932). Some supplementary researches on the collective risk theory, *Skandinavisk Aktuarietidskrift* **15**, 137–158.
- [31] Müller, A. & Pflug, G. (2001). Asymptotic ruin probabilities for risk processes with dependent increments, *Insurance: Mathematics and Economics* **28**, 381–392.
- [32] Paulsen, J. (1998). Ruin theory with compounding assets – a survey, *Insurance: Mathematics and Economics* **22**, 3–16.
- [33] Promislow, S.D. (1991). The probability of ruin in a process with dependent increments, *Insurance: Mathematics and Economics* **10**, 99–107.
- [34] Rolski, T., Schmidli, H., Schmidt, V. & Teugels, J. (1999). *Stochastic Processes for Insurance and Finance*, John Wiley & Sons, Chichester.
- [35] Ross, S. (1996). *Stochastic Processes*, 2nd Edition, Wiley, New York.
- [36] Schlegel, S. (1998). Ruin probabilities in perturbed risk models, *Insurance: Mathematics and Economics* **22**, 93–104.
- [37] Schmidli, H. (1995). Cramér–Lundberg approximations for ruin probabilities of risk processes perturbed by diffusion, *Insurance: Mathematics and Economics* **16**, 135–149.
- [38] Schmidli, H. (1996). Lundberg inequalities for a Cox model with a piecewise constant intensity, *Journal of Applied Probability* **33**, 196–210.
- [39] Schmidli, H. (1997). An extension to the renewal theorem and an application to risk theory, *Annals of Applied Probability* **7**, 121–133.
- [40] Sundt, B. & Teugels, J.L. (1995). Ruin estimates under interest force, *Insurance: Mathematics and Economics* **16**, 7–22.
- [41] Willmot, G.E. (1994). Refinements and distributional generalizations of Lundberg’s inequality, *Insurance: Mathematics and Economics* **15**, 49–63.
- [42] Willmot, G.E. (1996). A non-exponential generalization of an inequality arising in queueing and insurance risk, *Journal of Applied Probability* **33**, 176–183.
- [43] Willmot, G., Cai, J. & Lin, X.S. (2001). Lundberg inequalities for renewal equations, *Advances of Applied Probability* **33**, 674–689.
- [44] Willmot, G.E. & Lin, X.S. (1994). Lundberg bounds on the tails of compound distributions, *Journal of Applied Probability* **31**, 743–756.
- [45] Willmot, G.E. & Lin, X.S. (2001). *Lundberg Approximations for Compound Distributions with Insurance Applications*, Springer-Verlag, New York.
- [46] Yang, H. (1999). Non-exponential bounds for ruin probability with interest effect included, *Scandinavian Actuarial Journal* 66–79.

(See also **Collective Risk Theory; Time of Ruin**)

JUN CAI